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Inequalities for Polynomials with a Prescribed Zero

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If $P(z)$ is a polynomial of degree n, then according to a famous result known as Bernstein's inequality (see [3]),

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant n \operatorname*{Max}_{|z|=1} |P(z)|. \tag{1}$$

It was shown by Rahman and Mohammad [2] (see also [4, p. 370]) that if $P(1)=0$, then

$$\operatorname*{Max}_{|z|=1} |P(z)/(z-1)| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|. \tag{2}$$

O'Hara [1] has given a simple proof of the inequality (1) by using Lagrange's interpolation formula. In this paper we shall prove, by a new, simple, and analytical method, a sharpened form of the inequality (2). In lieu of requiring that the maximum of $|P(z)|$ on the right-hand side of (2) be taken on $|z|=1$, we only assume that it be taken over the nth roots of -1. The proof is based on the Cauchy–Schwarz inequality and the following lemma which is also of independent interest.

LEMMA. *Let $P(z)$ be a polynomial of degree n and $z_1, z_2, ..., z_n$ be the zeros of $z^n + a$, $a \neq 0$. Then for any complex number β such that $\beta^n + a \neq 0$, we have*

$$P'(\beta) = \frac{n\beta^{n-1}}{a+\beta^n} P(\beta) + \frac{a+\beta^n}{na} \sum_{k=1}^{n} P(z_k) \frac{z_k}{(z_k-\beta)^2} \tag{3}$$

and

$$\frac{1}{na} \sum_{k=1}^{n} \frac{z_k \beta}{(z_k-\beta)^2} = -\frac{n\beta^n}{(a+\beta^n)^2}. \tag{4}$$

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Proof. Consider the function

$$F(z)=\frac{P(z)-P(\beta)}{z-\beta},$$

which is a polynomial of degree $n-1$. Using Lagrange's interpolation formula with $z_1, z_2, \ldots, z_n$ as the basic points of interpolation we can write

$$F(z)=\sum_{k=1}^{n} F(z_k)\frac{z^n+a}{nz_k^{n-1}(z-z_k)}=\frac{1}{na}\sum_{k=1}^{n} F(z_k)\frac{z_k(z^n+a)}{(z_k-z)},$$

since $z_k^{n-1}=-a/z_k$. Now as $F(\beta)=P'(\beta)$, we get the identity in β:

$$P'(\beta)=\frac{a+\beta^n}{na}\sum_{k=1}^{n} F(z_k)\frac{z_k}{(z_k-\beta)}=\frac{a+\beta^n}{na}\sum_{k=1}^{n}(P(z_k)-P(\beta))\frac{z_k}{(z_k-\beta)^2}.$$

Equivalently,

$$P'(\beta)=\frac{a+\beta^n}{na}\sum_{k=1}^{n} P(z_k)\frac{z_k}{(z_k-\beta)^2}-\frac{a+\beta^n}{na}P(\beta)\sum_{k=1}^{n}\frac{z_k}{(z_k-\beta)^2}. \tag{5}$$

Taking, in particular, $P(z)=z^n$ in (5), we obtain

$$n\beta^{n-1}=-\frac{a+\beta^n}{na}\sum_{k=1}^{n}\frac{z_k}{(z_k-\beta)^2}.$$

Hence,

$$\frac{1}{na}\sum_{k=1}^{n}\frac{z_k}{(z_k-\beta)^2}=-\frac{n\beta^{n-1}}{(a+\beta^n)^2}. \tag{6}$$

Multiplying the two sides of (6) by β, we get (4). Using now (6) in (5), we obtain (3) and this completes the proof of the lemma.

We are now ready to prove the following

THEOREM. *Let $P(z)$ be a polynomial of degree n such that $P(1)=0$. If $z_1, z_2, \ldots, z_n$ are the zeros of z^n+1, then*

$$\operatorname*{Max}_{|z|=1}|P(z)/(z-1)|\leqslant\frac{n}{2}\operatorname*{Max}_{1\leqslant k\leqslant n}|P(z_k)|. \tag{7}$$

The result is best possible with equality in (7) for $P(z)=z^n-1$.

If $P(z)$ and $z_1, z_2, \ldots, z_n$ are as in the theorem, then $\operatorname{Max}_{|z|=1}|P(z)|$ may be larger than $\text{Max}_{1\leqslant k\leqslant n}|P(z_k)|$. As an example, for $n \geqslant 2$, consider the polynomial

$$\begin{aligned} P(z) &= z^n - 2z^{n/2} + 1 \qquad &\text{if} \quad n \text{ is even,} \\ &= z^n - 2z^{(n+1)/2} + 1 \qquad &\text{if} \quad n \text{ is odd.} \end{aligned}$$

Then

$$\underset{1\leqslant k\leqslant n}{\text{Max}} |P(z_k)| = 2 \qquad \text{in both the cases,}$$

whereas, for even $n \geqslant 2$,

$$\underset{|z|=1}{\text{Max}} |P(z)| = |P(e^{2\pi i/n})| = 4,$$

and for odd $n \geqslant 2$,

$$\underset{|z|=1}{\text{Max}} |P(z)| \geqslant |P(e^{2\pi i/n})| = |2(1+e^{i\pi/n})| = 4\cos(\pi/2n).$$

This example shows that for $n \geqslant 2$, the maximum of $|P(z)|$ on $|z| = 1$ may be at least $2\cos(\pi/2n)$ times as large as its maximum taken over the nth roots of -1, even if $P(1) = 0$.

Proof of the Theorem. Since $P(1) = 0$, $P(z)/(z-1)$ is a polynomial of degree $n-1$. If the maximum of $|P(z)/(z-1)|$ on $|z| = 1$ is attained at $z = z_k$ for some $k = 1, 2,..., n$, then

$$\begin{aligned} \underset{|z|=1}{\text{Max}} \left|\frac{P(z)}{z-1}\right| &= \left|\frac{P(z_k)}{z_k - 1}\right| \\ &\leqslant \frac{1}{|1-z_k|} \underset{1\leqslant k\leqslant n}{\text{Max}} |P(z_k)| \\ &\leqslant (n/2) \underset{1\leqslant k\leqslant n}{\text{Max}} |P(z_k)|, \end{aligned}$$

since $|1 - z_k| \geqslant 2/n$ for all $k = 1, 2,..., n$. Hence, the result follows in this case. Otherwise, suppose that the maximum of $|P(z)/(z-1)|$ on $|z| = 1$ is attained at $z = \beta$, where $|\beta| = 1$ and $\beta \neq z_k$ for all $k = 1, 2,..., n$, then

$$\underset{|z|=1}{\text{Max}} |P(z)/(z-1)| = |P(\beta)/(\beta - 1)|. \tag{8}$$

By Lagrange's formula with $z_1, z_2,..., z_n$ as the basic points of interpolation, we can write

$$\frac{P(z)}{z-1}=\sum_{k=1}^{n}\frac{P(z_k)}{(z_k-1)}\left(\frac{z^n+1}{nz_k^{n-1}(z-z_k)}\right)=\frac{1}{n}\sum_{k=1}^{n}\frac{P(z_k)(z^n+1)\,z_k}{(z_k-1)(z_k-z)},$$

since $z_k^{n-1}=-1/z_k$.

Therefore, from (8) we have

$$\begin{aligned}
&\left\{\operatorname*{Max}_{|z|=1}\left|\frac{P(z)}{z-1}\right|\right\}^2\\
&\quad=\left\{\left|\frac{P(\beta)}{\beta-1}\right|\right\}^2\\
&\quad\leqslant\left\{\frac{1}{n}\sum_{k=1}^{n}\left|\frac{z_k}{z_k-1}\right|\left|\frac{\beta^n+1}{z_k-\beta}\right|\right\}^2\left\{\operatorname*{Max}_{1\leqslant k\leqslant n}|P(z_k)|\right\}^2\\
&\quad\leqslant 1/n^2\left\{\sum_{k=1}^{n}\left|\frac{z_k}{z_k-1}\right|^2\right\}\left\{\sum_{k=1}^{n}\left|\frac{1+\beta^n}{z_k-\beta}\right|^2\right\}\left\{\operatorname*{Max}_{1\leqslant k\leqslant n}|P(z_k)|\right\}^2, \qquad (9)
\end{aligned}$$

by the Cauchy–Schwarz inequality.

Using now the 2nd part of the lemma with $a=1$, $\beta\neq z_k$, we get

$$\frac{1}{n}\sum_{k=1}^{n}\frac{z_k\beta}{(z_k-\beta)^2}=-\frac{n\beta^n}{(1+\beta^n)^2}.$$

Now, if $|z|=1$, $|\beta|=1$, and $z\neq\beta$ then $z\beta/(z-\beta)^2$ is a negative real number. In fact, if $z=e^{i\theta}$, $\beta=e^{i\alpha}$, then

$$\frac{e^{i(\theta+\alpha)}}{(e^{i\theta}-e^{i\alpha})^2}=-\frac{1}{4\sin^2(\theta-\alpha)/2}\qquad\text{provided}\quad\theta\not\equiv\alpha\ (\mathrm{mod}\ 2\pi).$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{|z_k-\beta|^2}&=\sum_{k=1}^{n}\frac{|z_k\beta|}{|z_k-\beta|^2}=-\sum_{k=1}^{n}\frac{z_k\beta}{(z_k-\beta)^2}\\
&=\frac{n^2\beta^n}{(1+\beta^n)^2}=n^2\left|\frac{\beta^n}{(1+\beta^n)^2}\right|\\
&=\frac{n^2}{|1+\beta^n|^2}. \qquad (10)
\end{aligned}$$

This gives, for $\beta=1$,

$$\sum_{k=1}^{n}\frac{1}{|z_k-1|^2}=-\sum_{k=1}^{n}\frac{z_k}{(z_k-1)^2}=\frac{n^2}{4}. \qquad (11)$$

Using (10) and (11) in (9), we obtain

$$(\operatorname*{Max}_{|z|=1} |P(z)/(z-1)|)^2 \leqslant (1/n^2)(n^2/4)\, n^2 (\operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|)^2.$$

Hence,

$$\operatorname*{Max}_{|z|=1} |P(z)/(z-1)| \leqslant \frac{n}{2} \operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|$$

which is (7), and the theorem is completely proved.

COROLLARY. *Let $P(z)$ be a polynomial of degree n such that $P(\beta)=0$, where β is an arbitrary nonnegative real number. If $z_1, z_2,..., z_n$ are the zeros of z^n+1, then*

$$\operatorname*{Max}_{|z|=1} |P(z)/(z-\beta)| \leqslant \frac{n}{1+\beta} \operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|. \tag{12}$$

Proof of the Corollary. Since $P(\beta)=0$, we can write $P(z)=(z-\beta)\,Q(z)$, where $Q(z)$ is a polynomial of degree $n-1$. Set $P*(z)=(z-1)\,Q(z)$; then

$$|P*(z_k)/P(z_k)| = |(z_k-1)/(z_k-\beta)| \leqslant 2/(1+\beta).$$

This gives

$$\operatorname*{Max}_{1\leqslant k\leqslant n} |P*(z_k)| \leqslant (2/(1+\beta)) \operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|. \tag{13}$$

Hence, from the above theorem and inequality (13) we get

$$\operatorname*{Max}_{|z|=1} |P(z)/(z-\beta)| = \operatorname*{Max}_{|z|=1} |P*(z)/(z-1)|$$

$$\leqslant \frac{n}{2} \operatorname*{Max}_{1\leqslant k\leqslant n} |P*(z_k)| \leqslant \frac{n}{1+\beta} \operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|,$$

which is the desired result.

Remark 1. Letting $z\to\beta$ in (12), we obtain

$$|P'(\beta)| \leqslant \frac{n}{1+\beta} \operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|, \tag{14}$$

for $0\leqslant\beta\leqslant 1$. For $\beta=1$, (14) is sharp.

Remark 2. Let $P(z)$ be a polynomial of degree n. Then

$$|P'(0)| \leqslant \operatorname*{Max}_{1\leqslant k\leqslant n} |P(z_k)|,$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + 1$. The result is best possible as shown by the polynomial $P(z) = z^n + z + 1$. This follows by taking $\alpha = 1$ and $\beta = 0$ in (3).

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